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A NOTE ON THE CHEREPANOV CALCULATION OF VISCOELASTIC FRACTURE

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RECENTLY Cherepanov [1, 2] has discussed crack propagation in continuous media on the basis of the thermodynamic power balance. The breadth of his treatment for many types of materials provides an excellent example of this method of analysing fracture and deserves careful study. In this note we only wish to comment upon one small segment [1] of his general treatment in which there appears to be some confusion in the examples presented for fracture in a 3-element model of a linear viscoelastic material (LVEM). In his Part 4, Cherepanov derives the necessary condition for limiting equilibrium for a finite length crack under conditions of plane strain and an incompressible LVEM. In terms of the stress intensity factor $N(t)$, which would equal $p^2 l/2$ for the usual Griffith problem, it is (Ref. [4.9]):

$$NE^{-1}N = 4\gamma/3\pi \tag{1}$$

where in general N is found from crack tip stress conditions and stress boundary conditions:

$$\sigma_x + \sigma_y = 2Nr^{-\frac{1}{2}}\cos(\theta/2) \tag{2}$$

in the usual way. E^{-1} is an operator defined by (Ref. [4.2]):

$$E^{-1}f(t) = \int_0^t E_0(t-\xi)f(\xi)\,\mathrm{d}\xi \tag{3}$$

where E_0 is a creep function to be given for each material and γ is the cohesive fracture energy.

Having deduced the governing integro–differential equation (1) he then proceeded to deduce $N(t)$ for various finite element models of a LVEM whose stress–strain law is represented by (Ref. [4.7]):

$$\varepsilon + \zeta\dot{\varepsilon} = (3\mu)^{-1}\dot{\sigma} + (3\eta)^{-1}\sigma. \tag{4}$$

There appears to be a slight error in the solution which prevents a clear physical interpretation of his results. Consider therefore his simplest example—for a Kelvin solid with the constitutive relation (Ref. [4.15]):

$$\sigma = 3\eta\dot{\varepsilon} + 3\mu\varepsilon \tag{5}$$

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for which the corresponding form of (1), using (3) is:

$$N(t)\int_0^t \frac{1}{3\eta}N(\xi)\exp[-\zeta(t-\xi)]\,d\xi = 4\gamma/3\pi \tag{6}$$

where $\zeta = \mu/\eta$. The solution of this equation is obtained through the intermediate step of reducing it to a differential equation, with the solution (Ref. [4.16]):

$$\frac{N^2(t)}{N_0^2} = \frac{(4\gamma/\mu\pi)-N^2(t)}{(4\gamma/\mu\pi)-N_0^2}\exp[2\zeta(t-t_0)] \tag{7}$$

outside of an apparent misprint of the factor 2 in the exponential term $N_0 = N(t_0)$ and t_0 are arbitrary constants.

Since we are only interested in positive $N(t)$, which is essentially the product of the applied fracture stress and crack size, the solution of (6) gives:

$$\frac{N(t)}{N_*} = \left\{1-\left[1-\frac{N_*^2}{N_0^2}\right]\exp[-2\zeta(t-t_0)]\right\}^{-\frac{1}{2}} \tag{8}$$

in which:

$$N_*^2 = 4\gamma\mu/\pi = 4\gamma E/3\pi \tag{9}$$

recognized as the critical Griffith stress to cause instability.

This expression (8) has two branches which Cherepanov plots for the case of the general LVEM (3-element) solid; one branch for $N_0 > N_*$ and the other for $N_0 < N_*$. Cherepanov then presents a physical interpretation of these two cases. While (8) satisfies the differential equation derived from (6) and also the condition $N = N_0$ at $t = t_0$, these latter constants do not appear to have been identified physically. If we now substitute (8) back into (6), it will be a solution provided:

$$N_0^2 = [N_0^2 - N_*^2]\exp 2\zeta t_0 \tag{10}$$

which means that if $N_0 < N_*$ (the Griffith critical stress), then the time must be imaginary. In other words, the branch of the solution for $N_0 < N_*$ is not a solution of (6) for real t_0. However, if $N_0 > N_*$ a real solution will result as might be expected, i.e. the applied stress exceeds the critical Griffith stress (based upon the minimum long time rubbery modulus, E_e).

Although the above argument gives a possible interpretation of N_0 and t_0 it is based on equation (6) being true for all time from $t = 0$. This implies from a physical point of view that fracture has begun at time $t = 0$ and (6) then describes the energy balance as the crack continues to grow. Another way of viewing the problem which is discussed more fully later is the following. The thermodynamically permissible relation:

$$\frac{N(t)}{3\eta}\int_0^t N(\tau)\exp(-\zeta(t-\tau))d\tau \leq \frac{4\gamma}{3\pi} \tag{11}$$

is such that the crack will not grow if $t < t_f$, where inequality holds; if the equality holds at $t = t_f$, t_f is the time to fracture. On the other hand if the equality does hold, then $t = t_f$, and t_f can be determined implicitly from (11). For times larger than t_f, i.e. $t > t_f$, after

fracture, then the balance equation must be written:

$$\frac{N(t)}{3\eta}\left\{\int_0^{t_f} N(\tau)\exp[-\zeta(t-\tau)]\,d\tau+\int_{t_f}^{t} N(\tau)\exp[-\zeta(t-\tau)]\,d\tau\right\} = 4\gamma/3\pi \tag{12}$$

where $N(\tau)$, $\tau < t_f$ and $N(t_f)$ are prescribed by the loading up to fracture and equation (12) determines $N(\tau)$, usually the crack length in terms of the applied loading, for $t > t_f$.

Solving (12), together with the fracture initiation condition of (11) at $t = t_f$, gives:

$$\frac{N(t)}{N_*} = \left\{1-\left[1-\frac{N_*^2}{N_f^2}\right]\exp[-2\zeta(t-t_f)]\right\}^{-\frac{1}{2}} \tag{13}$$

where $N_f = N(t_f)$. Equation (13) has two possible branches depending on whether $N_f^2 > N_*^2$ or $N_f^2 < N_*^2$ in exactly the same way as considered by Cherepanov. So we have now given an argument in which the N_0 and t_0 considered by Cherepanov could be associated with N_f and t_f as defined here. To test whether both these conditions are possible is more difficult than before, where we effectively considered the case $t_f = 0$. It is clear that solutions with $N_f^2 > 4\gamma\mu/\pi$ are possible since N equal to a constant in (11) gives an $N_f^2 > N_*^2$. For $N_f^2 < N_*^2$ however, one can show that there can exist no function $N(\tau)$ for which (11) is true with the equality sign at $t = t_f$ without violating the inequality condition of (11) for times $t < t_f$. So we have proved for this more sophisticated case that no solution to (12) with conditions (11) exists if $N_f^2 < N_*^2$.

Because the reference time and intensity factor did appear originally without much further designation, it is constructive to re-examine the fundamental equation for criticality, i.e. (1). It is possible to do so in a rather general sense, not restricted to a particular n-element model, as was used by Williams [3] in his thermodynamic power balance analysis of the spherical or cylindrical flaw model of a crack in a LVEM. Writing (1) in the equivalent form:

$$N(t)\left\{D_g N(t)+\int_0^t \frac{\partial D_{crp}(t-\xi)}{\partial(t-\xi)}N(\xi)\,d\xi\right\} = 4\gamma/3\pi \tag{14}$$

wherein $D_{crp}(t)$ is the creep compliance and $D_{crp}(0) \equiv D_g$ by definition. Equation (14) is thermodynamically permissible, up to the fracture time t_f, but in particular holds at the fracture initiation time t_f. In this way, one recognizes that of the two physical parameters making up the stress intensity factor for the assumed stress boundary value, i.e. applied stress $p(t)$ and the crack length $2l(t)$, ($N^2(t) = p^2(t)l(t)/2$ Ref. [4.17]), the crack length is of constant length $l = l_0$ for $t < t_f$. One can now proceed to determine t_f for a specified loading in a LVEM. The simplest example is the case of a step-function loading $p(t) = p_0 H(t)$ such that N is also a constant which as $t \to t_f$ becomes the critical value N_{cr}. Thus from (10):

$$N_{cr}(t_f)\left\{D_g N_{cr}(t_f)+N_{cr}\int_0^{t_f}\frac{\partial D_{crp}(t_f-\xi)}{\partial(t_f-\xi)}\,d\xi\right\} = 4\gamma/3\pi \tag{15}$$

or

$$N_{cr}(t_f) = \sqrt{\left[\frac{4\gamma/3\pi}{D_{crp}(t_f)}\right]} = p_{cr}(t_f)\sqrt{(l_0/2)} \tag{16}$$

as the implicit equation defining the time required to fracture a finite length crack, $2l_0$, subjected to a uniform stress p_{cr} in a LVEM characterized by a creep compliance $D_{crp}(t)$. As in the spherical-cylindrical flaw case, no fracture is possible in finite time if:

$$N_{cr}(t \to \infty) < \sqrt{\left[\frac{4\gamma/3\pi}{D_{crp}(\infty)}\right]} = \sqrt{\left(\frac{4E_e\gamma}{3\pi}\right)} \equiv N_* \tag{17}$$

as described in conjunction with (9), in which the long time, rubbery modulus is defined through $D_e = E_e^{-1}$. Indeed, the entire range of possible fracture initiation times $0 \leq t_f \leq \infty$ fall within the range controlled by $D_g \leq D_{crp}(t_f) \leq D_e$. In this constant applied stress example,

$$p_{cr_g} \equiv p_{cr}(t_f \to 0) = \sqrt{\left(\frac{8}{3\pi}\frac{E_g\gamma}{l_0}\right)}; \quad \text{brittle Griffith limit} \tag{18a}$$

$$p_{cr_e} \equiv p_{cr}(t_f \to \infty) = \sqrt{\left(\frac{8}{3\pi}\frac{E_e\gamma}{l_0}\right)}; \quad \text{rubbery Griffith limit} \tag{18b}$$

where this crack initiation time, t_f, could now be associated with the Cherepanov value of t_0 if desired.

Having now identified this crack initiation time, t_f, which if one desired could now be identified with the Cherepanov t_0 value (and prior to which no physically realizable crack velocity is possible—the $\dot{a}(t) = 0$ solution in Ref. [37]), it is possible to evaluate the crack velocity *after initiation* $t \geq t_f$ by further application of (14), i.e.

$$N(t)\left\{D_gN(t) + \int_0^{t_f} \frac{\partial D_{crp}(t-\xi)}{\partial(t-\xi)} N(\xi)\,\mathrm{d}\xi + \int_{t_f}^{t} \frac{\partial D_{crp}(t-\xi)}{\partial(t-\xi)} N(\xi)\,\mathrm{d}\xi\right\} = 4\gamma/3\pi \tag{19}$$

which is now the determining equation for $N(t)$ following the initiation. For example, and pursuing the same problem as before with say $p_{cr_g} < p_{cr}(t > t_f) < p_{cr_e}$, but now with a change in crack size $l(t > t_f)$, (19) becomes an integro–differential equation for determining $a(t)$ and subsequently the crack velocity $\dot{a}(t)$.

While we have been unable to solve for $a(t)$ in closed form for an arbitrary LVEM, use of the same 3-element model as Cherepanov's leads directly to the conclusion that no stable velocity of crack propagation after fracture initiation is possible for a uniform stress (as also deduced by Cherepanov), but further, can exist only for monotonic decreasing loads after fracture. In particular, one can deduce the required decay in applied loading after initiation in order to produce a stable or steady state crack propagation velocity, e.g. finding λ in the loading function in such physically interesting cases as [4].

$$p(t) = p_0(t/t_f) : 0 \leq t \leq t_f \tag{20a}$$

$$= p_0[-\lambda(t-t_f)] : t \geq t_f \tag{20b}$$

in order that $\dot{a}(t > t_f)$ is finite.

It is not the intent in this short note to elaborate upon these latter points, but a final important one does deserve mention. The basic paper [1] as well as much related research [2, 3] has been developed assuming that the specific fracture energy, γ, is time independent. Because this assumption, usually involved for analytical simplicity, is apparently far from true [5], it would be well for further extensions of fracture theory to account explicitly for this time-temperature dependence.

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